LECTURE (1)
CALCULUS 2

الدوال الزائدية

Hyperbolic Functions

المصادر: CALCULUS I

CALCULUS II

درسنا في السمستر الماضي الدوال المثلثية ومشتقاتها وكذالك الدوال المثلثية العكسية ومشتقاتها ايضاً. في هذا الدرس سوف نتطرق الى الدوال الزائدية والدوال الزائدية العكسية ومشتقاتها .

Definitions:

1- Hyperbolic sine of x : sinh x = $\frac{e^x - e^{-x}}{2}$.

2- Hyperbolic cosine of x : cosh x = $\frac{e^x + e^{-x}}{2}$.

3- Hyperbolic tangent of x : tanh x = $\frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

4- Hyperbolic cotangent of x : coth x = $\frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

3- Hyperbolic secant of x : sech x = $\frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$.

4- Hyperbolic cosecant of x : csch x = $\frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$.

Identities:

1- $cosh^2x - sinh^2x = 1$

2- $tanh^2x$ = 1- $sech^2x$

3- $coth^2x$ = 1+ $csch^2x$

4- $sinh^2x = \frac{\cosh 2x - 1}{2}$

5- $cosh^2x = \frac{\cosh 2x + 1}{2}$

6- $\sinh 2x = 2 \sinh x \, \cosh x$

7- $\cosh 2x = cosh^2x + sinh^2x$

8- cosh(-x) = cosh x and sinh(-x) = - sinh x

9- cosh x+ sinh x = e^x

solution:

cosh x+ sinh x = $\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}=\frac{e^x+e^{-x}+e^x-e^{-x}}{2}=e^x$

10- cosh x - sinh x=e^{-x}

solution:

cosh x - sinh x = $\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}=\frac{e^x+e^{-x}-e^x+e^{-x}}{2}=e^{-x}$

11- sinh(x+y)= sinh x cosh y + sinh y cosh x

12- cosh(x+y)= cosh x coshy + sinh x sinh y

Derivatives:

1- $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

2- $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$

3- $\frac{d}{dx}(\tanh u) = sech^2 u \frac{du}{dx}$

موجب

4- $\frac{d}{dx}(\coth u) =$ - $csch^2 u \frac{du}{dx}$

5- $\frac{d}{dx}(\text{sech}\, u) =$ - sech u tanh u $\frac{du}{dx}$

6- $\frac{d}{dx}(\text{csch}\, u) =$ - csch u coth u $\frac{du}{dx}$

سالب

Example 1: Find the derivative of $y = \sinh 3x$.

Solution: $\frac{dy}{dx} = y' = \cosh(3x) \cdot (3) = 3\cosh(3x)$

مشتقة الزاوية

Example 2: Find the derivative of $y = \tanh(1 + e^{2x})$

Solution: $\frac{dy}{dx} = y' = sech^2(1 + e^{2x})(2e^{2x}) = 2e^{2x}\, sech^2(1 + e^{2x})$

مشتقة الزاوية

Example 3: Find the derivative of $y = \cosh(\ln(x))$

Solution: $\frac{dy}{dx} = y' = \sinh(\ln x) \cdot \frac{1}{x} = \frac{\sinh(\ln x)}{x}$

مشتقة الزاوية

Example 4: Find the derivative of $y = sinh^3(5x)$.

نعتبر الدالة كأنما قوس مرفوع الى اس 3 ثم نشتق

Solution: $\frac{dy}{dx} = y' = 3.\ sinh^2(5x).(Cosh(5x).\ 5) = 15\ sinh^2(5x)Cosh(5x)$

(اس القوس) (القوس مطروح منه واحد) (مشتقة داخل القوس)

Example 5: Find the derivative of $y = sech\left(\frac{1}{x}\right)$

Solution: $\frac{dy}{dx} = y' = -\ sech\left(\frac{1}{x}\right) tanh\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \left(\frac{1}{x^2}\right)\ sech\left(\frac{1}{x}\right) tanh\left(\frac{1}{x}\right)$

LECTURE (2)

CALCULUS 2

الدوال الزائدية العكسية

The Inverses of the Hyperbolic Functions

المصادر: CALCULUS I

CALCULUS II

الدوال المثلثية الزائدية

Derivatives: مشتقات الدوال المثلثية الزائدية العكسية

1- $\frac{d(\sinh^{-1}u)}{dx} = \frac{du/dx}{\sqrt{1+u^2}}$

2- $\frac{d(\cosh^{-1}u)}{dx} = \frac{du/dx}{\sqrt{u^2-1}}, \quad u > 1$

3- $\frac{d(\tanh^{-1}u)}{dx} = \frac{du/dx}{1-u^2}, \quad |u| < 1$

4- $\frac{d(\coth^{-1}u)}{dx} = \frac{du/dx}{1-u^2}, \quad |u| > 1$

5- $\frac{d(\operatorname{sech}^{-1}u)}{dx} = -\frac{du/dx}{u\sqrt{1-u^2}}, 0 < u < 1$

6- $\frac{d(\operatorname{csch}^{-1}u)}{dx} = -\frac{du/dx}{|u|\sqrt{1+u^2}}, u \neq 0.$

Example 1: Find the derivative of y= $sinh^{-1}$ (2x).

Solution: $\frac{dy}{dx} = y' = \frac{2}{\sqrt{1+(2x)^2}} = \frac{2}{\sqrt{1+4x^2}}$

Example 2: Find the derivative of y= (1-x) $tanh^{-1}$ x .

Solution: $\frac{dy}{dx} = y' = (1-x)\frac{1}{1-x^2} + tanh^{-1} x \,.\, (-1)$

$= \frac{(1-x)}{(1-x)(1+x)} - tanh^{-1} x$

$= \frac{1}{(1+x)} - tanh^{-1} x$

Example 3: Find the derivative of y= $cosh^{-1}(\sec x)$

Solution: $\frac{dy}{dx} = \frac{dy}{dx} = y' = \frac{\sec x \tan x}{\sqrt{sec^2 x - 1}} = \frac{\sec x \tan x}{\sqrt{tan^2 x}} = \frac{\sec x \tan x}{\tan x} = \sec x.$

Exercises : Find $\frac{dy}{dx}$ in Exercises (1) - (6)

(1) $y = ln(\operatorname{sech} x)$.

(2) $y = 2 \tanh \frac{x}{2}$.

(3) $y = ln(\operatorname{csch} x + \coth x)$.

(4) $y = tanh^{-1}(\sin x)$.

(5) $y = x^2 \, csch^{-1} \, (x^2)$.

(6) $y = (1 - x^2) \, coth^{-1} x$

LECTURE (3)

CALCULUS 2

التكامل غير المحدد

Indefinite Integral

المصادر: CALCULUS I

CALCULUS II

التكامل

Definition: (Indefinite Integral) التكامل غير المحدد

The set all antiderivatives of *f(x)* is called indefinite integral of f with respect to x and denoted by:

$$\int \quad f(x)dx \quad = \quad F(x) + c$$

اشارة التكامل دالة التكامل ثابت التكامل

Basic Integration Formulas

1. $\int k\, f(x)du = k \int f(u)du$
2. $\int [f(u) \pm g(x)]\, du = \int f(u)du \pm \int g(u)du$
3. $\int du = u + c$
4. $\int u^n\, du = \frac{u^{n+1}}{n+1} + c\,, n \neq -1$
5. $\int \frac{du}{u} = ln|u| + c$
6. $\int e^u du = e^u + c$

7. $\int \sin u \, du = -\cos u + c$

8. $\int \cos u \, du = \sin u + c$

9. $\int \tan u \, du = -ln|\cos u| + c$

10 . $\int \cot u \, du = ln|\sin u \,| + c$

11. $\int \sec u \, du = ln|\sec u + \tan u| + c$

12. $\int \csc u \, du = -ln|\csc u + \cot u| + c$

13. $\int sec^2 u \, du = \tan u + c$

14. $\int csc^2 u \, du = -\cot u + c$

15. $\int \sec u \tan u \, du = \sec u + c$

16. $\int \csc u \cot u \, du = -\csc u + c$

17. $\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1} \frac{u}{a} + c$

18. $\int \frac{du}{a^2+u^2} = \frac{1}{a} tan^{-1} \frac{u}{a} + c$

19. $\int \frac{du}{\sqrt{u^2-a^2}} = \frac{1}{a} sec^{-1} \frac{|u|}{a} + c$

طرق التكامل

الطريقة الاولى: التكامل المباشر (اي نستطيع ان نكامل لتوفر المشتقة).

Example (1): Evaluate the integral $\int (3x^4 + x^{\frac{2}{3}} - 2\,x^{-4} - \sqrt{2})\,\mathrm{dx}$

Solution: $\int (3x^4 + x^{\frac{2}{3}} - 2\,x^{-4} - \sqrt{2})\,\mathrm{dx}$

$$= \ 3\frac{x^5}{5} + \frac{x^{\frac{5}{3}}}{\frac{5}{3}} - 2\ \frac{x^{-3}}{-3} - \sqrt{2}\ x + c.$$

$$= \frac{3}{5}x^5 + \frac{3}{5}x^{\frac{5}{3}} + \ \frac{2}{3x^3} - \sqrt{2}\ x + c.$$

Example (2): Evaluate the integral $\int \sqrt{(3x-1)^3}dx$

Solution: $\int \sqrt{(3x-1)^3}\,\mathrm{dx} = \int \frac{3}{3}(3x-1)^{\frac{3}{2}}\,dx$

$$= \frac{1}{3}\frac{(3x-1)^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{15}(3x-1)^{\frac{5}{2}} + c$$

Example (3): Evaluate the integral $\int (x^2+1)^3\ x\ dx$

Solution: $\int \frac{2}{2}(x^2+1)^3\,dx = \frac{1}{2}\frac{(x^2+1)^4}{4} + c = \frac{1}{8}(x^2+1)^4 + c$

Example (4): Evaluate the integral $\int \frac{\sqrt{1+\tan x}}{cos^2 x} dx$

Solution: $\int \frac{\sqrt{1+\tan x}}{cos^2 x} dx = \int \sqrt{1+\tan x}\ \ sec^2 x$ dx

$= \int (1+\tan x)^{\frac{1}{2}}\ sec^2 x$ dx

$= \frac{(1+\tan x)^{\frac{3}{2}}}{\frac{3}{2}}$ + c

Example (5): Evaluate the integral $\int \sqrt{1+\sin 2x}\ dx$

Solution: $\int \sqrt{1+\sin 2x}\ dx = \int \sqrt{(cos^2 x + sin^2 x) + (2 \sin x\ cos x)}$ dx

$= \int \sqrt{cos^2 x + 2 \sin x\ cos x\ + sin^2 x}$ dx $= \int \sqrt{(\cos x + \sin x)^2}$ dx

$= \int (\cos x + \sin x)$ dx = sin x – cos x +c

- Exercises : Evaluate the Following Integrals

1. $\int \frac{dx}{(a+bx)^{\frac{1}{3}}}$.
2. $\int \ln(\sin x) \cot x \, dx$.
3. $\int \frac{dx}{(x+2\sqrt{x}+1)\sqrt{x}}$.
4. $\int \sqrt{1+\cos 6x}$ dx.
5. $\int \frac{e^{\frac{1}{x}}}{x^2} \, dx$.
6. $\int (x^6 + 6x^3 + 9)^5 x^2 \, dx$.

LECTURE (4)
CALCULUS 2

Definite Integral

and

Integration By Substitution

المصادر: CALCULUS I

CALCULUS II

(Definite Integral) التكامل المحدد

The integral $\int_a^b f(x)dx$ is called the definite integral of the function *f(x)* over the interval $[a, b]$.

Properties of definite integral   خواص التكامل المحدد

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(2) $\int_a^a f(x)dx = 0$.

(3) $\int_a^b k\, f(x)dx = k \int_a^b f(x)dx$.

(4) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

(5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a, b]$.

Example (1): Evaluate the integral $\int_{-3}^{2}(6-x-x^2)$ dx

Solution: $\int_{-3}^{2}(6-x-x^2)$dx=$\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$

$= \left[6(2)-\frac{(2)^2}{2}-\frac{(2)^3}{3}\right]-\left[6(-3)-\frac{(-3)^2}{2}-\frac{(-3)^3}{3}\right]$

$= \left[12-2-\frac{8}{3}\right]$-$\left[-18-\frac{9}{2}+9\right]$= $\left[10-\frac{8}{3}\right]$-$\left[-9-\frac{9}{2}\right]$

$= 19-\frac{8}{3}+\frac{9}{2} = 19+\frac{-16+27}{6} = 19+\frac{11}{6} = \frac{125}{6}$.

Example (2): Evaluate the integral $\int_0^{\pi} \sin x \, dx$

Solution: $\int_0^{\pi} \sin x \, dx = -\cos x|_0^{\pi} = -(\cos \pi - \cos o)$

= -(-1-1)= -(-2)=2

Example (3): Evaluate the integral $\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{cos^2 x} dx$

Solution: $\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{cos^2 2x} dx = \int_0^{\frac{\pi}{6}} (\cos 2x)^{-2} \sin 2x \; dx$

$= \left(\frac{1}{-2}\right) \int_0^{\frac{\pi}{6}} (\cos 2x)^{-2} \quad (-2) \sin 2x \; dx$

$= \frac{-1}{2} \left(\frac{(\cos 2x)^{-1}}{-1}\right)_0^{\frac{\pi}{6}} = \frac{1}{2} \left(\frac{1}{\cos 2x}\right)_0^{\frac{\pi}{6}}$

$= \frac{1}{2} \left[\frac{1}{\cos\left(2 \frac{\pi}{6}\right)} - \frac{1}{\cos(2\,(0))}\right] = \frac{1}{2} \left[\frac{1}{\cos\left(\frac{\pi}{3}\right)} - \frac{1}{\cos(0)}\right] = \frac{1}{2} \left[\frac{1}{\frac{1}{2}} - \frac{1}{1}\right]$

$= \frac{1}{2} \; [2 - 1] = \frac{1}{2}$

Example (4): Evaluate the integral $\int_0^2 (x^3+2)^{\frac{1}{2}} x^2\, dx$

Solution: $\int_0^2 (x^3+2)^{\frac{1}{2}} x^2\, dx = \left(\frac{1}{3}\right) \int_0^2 (x^3+2)^{\frac{1}{2}}\ (3)x^2\, dx$

$$= \frac{1}{3}\ \frac{(x^3+2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^2 = \frac{2}{9}\ (x^3+2)^{\frac{3}{2}}\Big|_0^2 = \frac{2}{9}\left[\left((2)^3+2\right)^{\frac{3}{2}} - \left((0)^3+2\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{9}\left[(8+2)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] = \frac{2}{9}\left[(10)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] = \frac{2}{9}\left[\sqrt{(10)^3} - \sqrt{(2)^3}\right]$$

$$= \frac{2}{9}\left[\sqrt{1000} - \sqrt{8}\right] = \frac{2}{9}\left[10\sqrt{10} - 2\sqrt{2}\right].$$

الطريقة الثانية في التكامل

التكامل بالتعويض Integration By Substitution

تتمثل هذه الطريقة بتحويل التكامل $I = \int f\big(g(x)\big)g'(x)dx$ الى الصيغة :
$\int f(u)du$ و كمايلي .

1) تعويض $u=g(x)$ ثم نجد $du = g'(x)dx$.

2) نعوض عن قيمة **dx** بدلالة du و قيمة u بدلالة $g(x)$ لنحصل على الحل.

Example (1): Evaluate the integral $\int \frac{dx}{\sqrt[3]{1-2x}}$

Solution: $I = \int \frac{dx}{\sqrt[3]{1-2x}} = \int (1-2x)^{\frac{-1}{3}} \, dx$

Let $u=1-2x \rightarrow du= -2dx$

$\therefore dx = \frac{du}{-2}$

$$I = \int u^{\frac{-1}{3}} \frac{du}{-2} = \frac{-1}{2} \int u^{\frac{-1}{3}} \, du = \frac{-1}{2} \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{-3}{4} u^{\frac{2}{3}} + c$$

الان نعوض عن u بدلالة x

$$I = \frac{-3}{4} (1-2x)^{\frac{2}{3}} + c$$

Example (2): Evaluate the integral $I = \int sin^2 5x \, cos5x \, dx$

Solution: Let $u=sin5x \rightarrow du= 5cos5x \, dx$

$\therefore dx = \frac{du}{5cos5x}$

$$I = \int sin^2 5x \; cos5x \, dx = \int u^2 \frac{cos5x}{5cos5x} du = \int u^2 \frac{1}{5} du = \frac{1}{5} \frac{u^3}{3} + c = \frac{u^3}{15} + c$$

$$\therefore I = \frac{(sin5x)^3}{15} + c$$

Example (3): Evaluate the integral $\mathrm{I} = \int x \ e^{x^2+1} \, dx$

Solution: let $u = x^2 + 1 \rightarrow du = 2x \, dx \rightarrow dx = \frac{du}{2x}$

$$I = \int e^u x \frac{du}{2x} = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + c$$

$$I = \frac{1}{2} e^{x^2+1} + c$$

Example (4): Evaluate the integral $I = \int_0^1 (x^2 + 1) \ x \, dx$

Solution: let $u = x^2 + 1 \rightarrow du = 2x \, dx \rightarrow dx = \frac{du}{2x}$

at $x = 0 \rightarrow u = 1$

at $x = 1 \rightarrow u = 2$

$$\therefore \ I = \int_1^2 u \, \frac{1}{2} \, du = \frac{1}{2} \frac{u^2}{2} \Big|_1^2 = \frac{1}{4} (4 - 1) = \frac{3}{4}$$

LECTURE (5)

CALCULUS 2

Integration by Parts method

المصادر: CALCULUS I
CALCULUS II

التكامل بالتجزئة   :Integration by Parts

وهي في الاغلب تكون من تكاملات فيها حاصل ضرب دالتين ليس لاحدهما علاقة بالاخر ولها قانون عام ويعرف كالاتي.

$$\int u\,dv = uv - \int v\,du \;\rightarrow$$ القانون العام

في هذه الطريقة اول خطوة نعملها هو يجب تحديد اي الدالة نفرضها u وايهما نفرضها v

دائما نفرض الدالة القابلة للاشتقاق u والدالة القابلة للتكامل نفرضها v

Example (1): Evaluate the integral $\int \ln x\, dx$

Solution: let $u = \ln x \quad , \qquad dv = dx$

$$du = \frac{1}{x}\,dx\,, \qquad v = x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = \int \ln x\,dx = x\ln x - \int x\,.\frac{1}{x}\,dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

Example (2): Evaluate the integral $\int x \ln x \, dx$

Solution: let $u = \ln x \qquad , dv = x\,dx$

$$du = \frac{1}{x}\,dx \qquad , v = \frac{x^2}{2}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = \int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Example (3): Evaluate the integral $\int x\,e^x\,dx$

Solution: let $u = x \qquad , dv = e^x\,dx$

$$du = dx \qquad , v = e^x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = \int x\,e^x\,dx = x\,e^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

Example (4): Evaluate the integral $\int x^2 \; e^x \, dx$

Solution: let $u = x^2$, $dv = e^x \, dx$

$du = 2x$, $v = e^x$

$\int u \, dv = uv - \int v \, du$

$\int u \, dv = \int x^2 \; e^x \; dx = x^2 \, e^x - \int 2x \, e^x dx$ ... (1)

ايضا تكامل بالتجزئة لذلك نكامل مرة ثانية

let $u = 2x$, $dv = e^x \, dx$

$du = 2$, $v = e^x$

$\int u \, dv = uv - \int v \, du$

$\therefore \int 2x \, e^x dx = 2x \, e^x - \int 2 \, e^x dx$

$= 2x \, e^x - 2 \, e^x$ ... (2)

نعوض معادلة (2) في معادلة (1)

$\to \int x^2 \, e^x dx = x^2 \, e^x - (2x \, e^x - 2 \, e^x) + c$

$= x^2 \, e^x - 2x \, e^x + 2 \, e^x + c$

Tabular Integration

في بعض المسائل $\int f(x)g(x)dx$ تكون *f(x)* يمكن اشتقاقها عدد من المرات تودي الى اضحلالها اي $\left(\frac{d^n f(x)}{dx^n} = 0\right)$ بعد n من الاشتقاقات والدالة *g(x)* يمكن تكاملها لعدد من المرات فنتبع الطريقة التالية للحل من الجدول التالي.

f(x) ومشتقاتها		*g(x)* وتكاملاتها
$f(x)$		$g(x)$
$f'(x)$	$+$	$\int g(x)dx = g_1(x)$
$f''(x)$	$-$	$\int g_1(x)dx = g_2(x)$
$f'''(x)$	$+$	$\int g_2(x)dx = g_3(x)$
$\vdots$		$\vdots$
$f^{(n-1)}(x)$	$\pm$	
$f^{n}(x) = 0$		$\int g_{n-1}(x)dx = g_n(x)$

ويكون الناتج النهائي بالشكل:

$$\int f(x)g(x)dx = f(x).g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - \ldots (\pm)f^{(n-1)}(x)g_n(x) + c$$

Example (5): Back to Ex(4) $\int x^2 \; e^x \, dx$

Solution: let $f(x) = x^2$ *and* $g(x) = e^x$

	f(x) and its der.		g(x) and its int.	
اشتقاق	x^2	+	e^x	تكامل
	2x	−	e^x	
	2	+	e^x	
	0		e^x	

$$\int x^2 \; e^x \, dx = x^2 e^x - 2xe^x + 2e^x + c$$

Example (6): Evaluate the integral $\int (x^3 - 2x^2 + 3x + 1) \sin 2x \, dx$

Solution: let $f(x) = (x^3 - 2x^2 + 3x + 1)$ *and* $g(x) = \sin 2x$

f(x) and its der.		g(x) and its int.
$x^3 - 2x^2 + 3x + 1$	+	$\sin 2x$
$3x^2 - 4x + 3$	−	$\frac{-1}{2} \cos 2x$
6x-4	+	$\frac{-1}{4} \sin 2x$
6	−	$\frac{1}{8} \cos 2x$
0		$\frac{1}{16} \sin 2x$

$$\int (x^3 - 2x^2 + 3x + 1) \sin 2x \, dx = (x^3 - 2x^2 + 3x + 1)\left(\frac{-1}{2}\cos 2x\right) - (3x^2 - 4x + 3)\left(\frac{-1}{4}\sin 2x\right)$$

$$+ (6x-4)\left(\frac{1}{8}\cos 2x\right) - 6\left(\frac{1}{16}\sin 2x\right) + c$$

Example (7): Evaluate the integral $\int e^x \sin x \, dx$

Solution: let $u = e^x \quad , dv = \sin x \, dx$

$du = e^x dx \quad , v = -\cos x$

$\int u \, dv = uv - \int v \, du$

$\int u \, dv = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$ ... (1)

let $u = e^x \quad , dv = \cos x \, dx$

$du = e^x dx \quad , v = \sin x$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$ ...(2)

نعوض معادلة (2) في معادلة (1)

$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$

Exercises :

(1) $\int x^2 \ln(x+1)\ dx$

(2) $\int x\ (\ln x)^2\ \ dx.$

(3) $\int (x^3 + x^2 + x + 1)e^{-2x}\ dx.$

(4) $\int e^{-x} \sin x\ \ dx.$

(5) $\int_0^1 x\sqrt{1-x}\, dx\,.$

LECTURE (6)

CALCULUS 2

Certain Powers of Trigonometric
and
Hyperbolic Integrals

المصادر: CALCULUS I
CALCULUS II

Certain Powers of Trigonometric and Hyperbolic Integrals

Consider the following integrals form:

$$(A) \quad \int sin^m u \;\; cos^n u \, du \qquad or \qquad \int sinh^m u \;\; cosh^n u \, du$$

$$(B) \quad \int tan^m u \;\; sec^n u \, du \qquad or \qquad \int tanh^m u \;\; sech^n u \, du$$

$$(C) \quad \int cot^m u \;\; csc^n u \, du \qquad or \qquad \int coth^m u \;\; csch^n u \, du$$

Under Type *(A)* , there are three cases:

Case 1: If *m* is odd and positive integers , we factor out *sin u* *(sinh u)* and change the remaining even power of *sin u* *(sinh u)* to *cos u (cosh u)* using the identities:

$$\sin^2 u = 1 - \cos^2 u \quad , \quad \sinh^2 u = \cosh^2 u - 1$$

Example (1): Evaluate the integral $\int \sin^5 2x \, \cos^{\frac{-3}{2}} 2x \, dx$

Solution: $\int \sin^5 2x \, \cos^{\frac{-3}{2}} 2x \, dx = \int \sin^4 2x \, \cos^{\frac{-3}{2}} 2x \, \sin 2x \, dx$

$$= \int (1 - \cos^2 2x)^2 \cos^{\frac{-3}{2}} 2x \sin 2x \, dx = \int (1 - 2\cos^2 2x + \cos^4 2x) \cos^{\frac{-3}{2}} 2x \sin 2x \, dx$$

$$= \int \left(\cos^{\frac{-3}{2}} 2x - 2\cos^{\frac{1}{2}} 2x + \cos^{\frac{5}{2}} 2x \right) \sin 2x \, dx$$

$$= \int (\cos^{\frac{-3}{2}} 2x \, \sin 2x - 2\cos^{\frac{1}{2}} 2x \sin 2x + \cos^{\frac{5}{2}} 2x \sin 2x) \, dx$$

$$= \left[-\frac{1}{2} \frac{\cos^{\frac{-1}{2}} 2x}{\frac{-1}{2}} + \frac{\cos^{\frac{3}{2}} 2x}{\frac{3}{2}} + \frac{-1}{2} \frac{\cos^{\frac{7}{2}} 2x}{\frac{7}{2}} \right] + C = \cos^{\frac{-1}{2}} 2x + \frac{2}{3} \cos^{\frac{3}{2}} 2x - \frac{1}{7} \cos^{\frac{7}{2}} 2x + C$$

Case 2: If *n* is odd and positive integers , we factor out *cos u (cosh u)* and change the remaining even power of *cos u (cosh u)* to *sin u (sinh u)* using the identities:

$$\cos^2 u = 1 - \sin^2 u \quad , \quad \cosh^2 u = 1 + \sinh^2 u$$

Example (2): Evaluate the integral $\int \sinh^4 3x \ \cosh^3 3x \, dx$

Solution: $\int \sinh^4 3x \ \cosh^3 3x \, dx = \int \sinh^4 3x \ \cosh^2 3x \cosh 3x \ dx$

$$= \int \sinh^4 3x \, (1 + \sinh^2 3x) \cosh 3x \, dx$$

$$= \int (\sinh^4 3x \cosh 3x + \sinh^6 3x \cosh 3x \,) \, dx$$

$$= \left[\frac{1}{3}\frac{\sinh^5 3x}{5} + \frac{1}{3}\frac{\sinh^7 3x}{7}\right] + C = \frac{\sinh^5 3x}{15} + \frac{\sinh^7 3x}{21} + \text{C}$$

Case 3: If both *m* and *n* are even and positive integers (or one of them zero) , we reduce the degree of the expression by using the identities:

$$\sin^2 u = \frac{1-\cos 2u}{2} \quad , \quad \sinh^2 u = \frac{\cosh 2u - 1}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2} \quad , \quad \cosh^2 u = \frac{\cosh 2u + 1}{2}$$

Example (3): Evaluate the integral $\int \sin^2 2x \ \cos^2 2x \, dx$

Solution: $\int \sin^2 2x \ \cos^2 2x \, dx = \int \left(\frac{1-\cos 4x}{2}\right)\left(\frac{1+\cos 4x}{2}\right) dx$

$$= \frac{1}{4}\int (1 - \cos 4x)(1 + \cos 4x)\, dx$$

$$= \frac{1}{4}\int (1 - \cos^2 4x)dx = \frac{1}{4}\int \left(1 - \left(\frac{1+\cos 8x}{2}\right)\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 8x\right) dx = \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{4}\left[\frac{x}{2} - \frac{1}{16}\sin 8x\right] + C$$

Under Type *(B)* , there are two cases:

Case 1: If *n* is even and positive integers, we factor out $sec^2 u\ (sech^2 u)$ and change the remaining even power of *sec u* *(sech u)* to *tan u (tanh u)* using the identities:

$$sec^2 u = 1 + tan^2 u \quad , \quad sech^2 u = 1 - tanh^2 u$$

Example (4): Evaluate the integral $\int sech^4 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$

Solution: $\int sech^4 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx = \int sech^2 \frac{x}{2}\ sech^2 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$

$$= \int (1 - tanh^2 \frac{x}{2})\ sech^2 \frac{x}{2}\ tanh^{\frac{-1}{3}} \frac{x}{2} dx$$

$$= \int \left(tanh^{\frac{-1}{3}} \frac{x}{2} sech^2 \frac{x}{2} - tanh^{\frac{5}{3}} \frac{x}{2} sech^2 \frac{x}{2} \right) dx = \left[2 \frac{tanh^{\frac{2}{3}} \frac{x}{2}}{\frac{2}{3}} - 2 \frac{tanh^{\frac{8}{3}} \frac{x}{2}}{\frac{8}{3}} \right] + c$$

$$= \left[tanh^{\frac{2}{3}} \frac{x}{2} - \frac{3 tanh^{\frac{8}{3}} \frac{x}{2}}{4} \right] + c$$

Case 2: If *m* is odd and positive integers, we factor out *sec u tan u (sech u tanh u)* and change the remaining even power of *tan u (tanh u)* to *sec u (sech u)* using the identities:

$$tan^2 u = sec^2 u - 1 \qquad , \qquad tanh^2 u = 1 - sech^2 u$$

Example (2): Evaluate the integral $\int tan^3 2x \; sec^{\frac{-1}{4}} 2x \, dx$

Solution: $\int tan^3 2x \; sec^{\frac{-1}{4}} 2x \, dx = \int tan^2 2x \; sec^{\frac{-5}{4}} 2x \; (\tan 2x \; \sec 2x) \, dx$

$$= \int (sec^2 2x - 1) \, sec^{\frac{-5}{4}} 2x \; (\sec 2x \tan 2x) \, dx$$

$$= \int \left[sec^{\frac{3}{4}} 2x - sec^{\frac{-5}{4}} 2x\right] (\sec 2x \tan 2x) \, dx$$

$$= \int \left[sec^{\frac{3}{4}} 2x \, (\sec 2x \tan 2x) - sec^{\frac{-5}{4}} 2x (\sec 2x \tan 2x)\right] dx$$

$$= \left[\frac{1}{2} \frac{sec^{\frac{7}{4}} 2x}{\frac{7}{4}} - \frac{1}{2} \frac{sec^{\frac{-1}{4}} 2x}{\frac{-1}{4}}\right] + C = \frac{2}{7} \, sec^{\frac{7}{4}} 2x + 2 \, sec^{\frac{-1}{4}} 2x + C$$

Under Type *(C)* , there are two cases similar to those of type *(B)* where

The identities:

Case 1: $csc^2 u = 1 + cot^2 u$, $csch^2 u = coth^2 u - 1$

Case 2: $cot^2 u = csc^2 u - 1$, $coth^2 u = 1 + csch^2 u$

Exercises :

(1) $\int sin^5 2x\ dx$

(2) $\int cos^3 x\ \ cos^{\frac{-1}{2}} x\, dx.$

(3) $\int csc^6 x\ dx$.

(4) $\int tan^3 x\ \sec x\ dx.$

(5) $\int_0^1 sinh^4 x\, dx$.

LECTURE (7)

CALCULUS 2

Trigonometric Substitutions

المصادر: CALCULUS I
CALCULUS II

الطريقة الخامسة في التكامل

Trigonometric Substitutions: التعويضات المثلثية

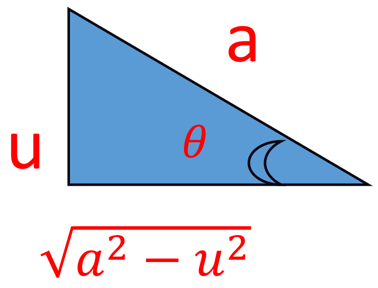
If the integral involve one of forms $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$, a^2-u^2, a^2+u^2 or u^2-a^2. Then the substitutions as follows:

(1) If $\sqrt{a^2-u^2}$, let $u = a\, sin\theta \;\rightarrow\; a^2-u^2 = a^2\, cos^2\theta$

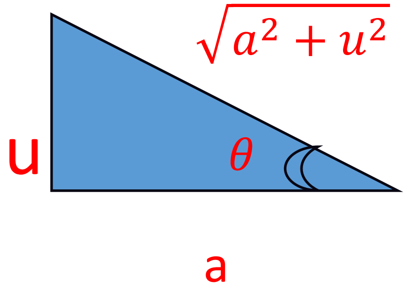
(2) If $\sqrt{a^2+u^2}$, let $u = a\, tan\theta \;\rightarrow\; a^2+u^2 = a^2\, sec^2\theta$

(3) If $\sqrt{u^2-a^2}$, let $u = a\, sec\theta \;\rightarrow\; u^2-a^2 = a^2\, tan^2\theta$

(1)



(2)



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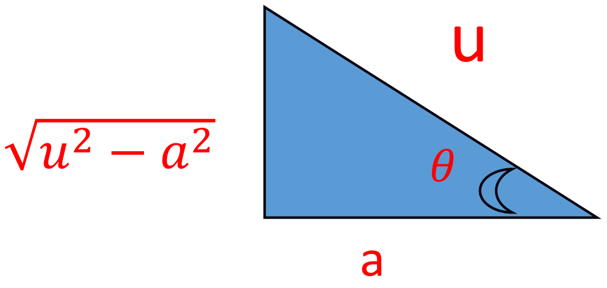


لايجاد القيم التي ظهرت على المثلثات الثلاثة نستخدم نظرية فيثاغورس والتي تنص :

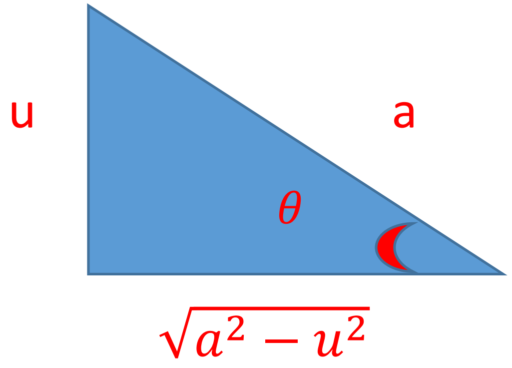
مربع الوتر = مربع المقابل + مربع المجاور

مثلاً في المثلث رقم (1)

$u = a\,sin\theta \rightarrow \qquad sin\theta = \frac{u}{a}$

المقابل

الوتر



$$(a)^2 = (u)^2 + (\text{المجاور})^2$$

$$(\text{المجاور})^2 = a^2 - u^2$$

$$\text{المجاور} = \sqrt{a^2 - u^2}$$

كذالك بالنسبة الى مثلث رقم (2) و(3)

Example (1): Evaluate the integral $\int \frac{dx}{4+x^2}$

Solution: method (1) من قانون تكامل الدوال العكسية

$$\int \frac{dx}{4+x^2} = \frac{1}{2} tan^{-1}\frac{x}{2} + C$$

method (2)

$$x = 2tan\theta \rightarrow tan\theta = \frac{x}{2} \rightarrow \theta = tan^{-1}\frac{x}{2}$$

$$dx = 2sec^2\theta\ d\theta$$

$$\int \frac{dx}{4+x^2} = \int \frac{2\ sec^2\theta\ d\theta}{4+4tan^2\theta} = \int \frac{2sec^2\theta\ d\theta}{4(1+tan^2\theta)} = \int \frac{2sec^2\theta\ d\theta}{4sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\ tan^{-1}\frac{x}{2} + C$$

Example (2): Evaluate the integral $\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}dx$

Solution: $x = sin\theta$ at $x = \frac{\sqrt{3}}{2} \to \frac{\sqrt{3}}{2} = sin\theta \to \theta = \frac{\pi}{3}$

$dx = cos\theta\ d\theta$ at $x = -\frac{1}{2} \to -\frac{1}{2} = sin\theta \to \theta = -\frac{\pi}{6}$

$$\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-sin^2\theta}\ cos\theta\ d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} cos^2\theta\ d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2} d\theta$$

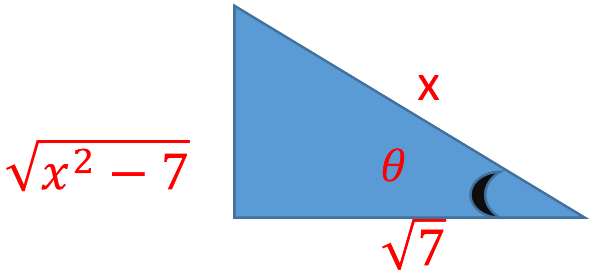
$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2}\left[\left(\frac{\pi}{3} + \frac{1}{2}\ sin\frac{2\pi}{3}\right) - \left(-\frac{\pi}{6} + \frac{1}{2}\ sin\left(\frac{-\pi}{3}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}\right] = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sqrt{3}}{2}\right] = \frac{\pi + \sqrt{3}}{4}$$

Example (3): Evaluate the integral $\int \frac{\sqrt{x^2-7}}{x}\, dx$

Solution: $x = \sqrt{7}sec\theta \ \rightarrow sec\theta = \frac{x}{\sqrt{7}} \rightarrow \ \theta = sec^{-1}\frac{x}{\sqrt{7}}$

$$dx = \sqrt{7}\, sec\theta\, tan\theta\, d\theta$$



$$\int \frac{\sqrt{x^2-7}}{x}\, dx = \int \frac{\sqrt{7sec^2\theta - 7}}{\sqrt{7}\, sec\theta} \sqrt{7}sec\theta\, tan\theta\, d\theta$$

$$= \int \frac{\sqrt{7}\, tan\theta}{\sqrt{7}\, sec\theta} \sqrt{7}sec\theta\, tan\theta\, d\theta = \int \sqrt{7}tan^2\theta\, d\theta$$

$$= \sqrt{7}\int (sec^2\theta - 1)d\theta = \sqrt{7}[tan\theta - \theta] + C$$

$$= \sqrt{7}\left[tan\left(sec^{-1}\frac{x}{\sqrt{7}}\right) - sec^{-1}\frac{x}{\sqrt{7}}\right] + C$$

$$or = \sqrt{7}\left[\frac{\sqrt{x^2-7}}{\sqrt{7}} - sec^{-1}\frac{x}{\sqrt{7}}\right] + C$$

Exercises :

(1) $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}}$.

(2) $\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$.

(3) $\int_0^2 \frac{x^2dx}{x^2+4}$.

(4) $\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$.

LECTURE (8)
CALCULUS 2

Integrals Involving Quadratic Functions

المصادر: CALCULUS I
CALCULUS II

Integrals Involving Quadratic Functions
تكاملات تتضمن دوال تربيعة

If the integral involve a quadratic function $x^2 + ax + b$, we reduced

It to the form $u^2 + B$ by completing the square as follows:

$$x^2 + ax + b = x^2 + ax + \frac{a^2}{4} + b - \frac{a^2}{4}$$

$$= \left(x + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$$

$$= u^2 + B$$

Then the solution can be found by Method [4] or [5].

Example (1): Evaluate the integral $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution: $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x)}} = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{-[(x-1)^2-1]}}$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

Let $u = x - 1 \quad \rightarrow \qquad du = dx$

$$\int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \frac{du}{\sqrt{1-u^2}} = sin^{-1}u + C = sin^{-1}(x-1) + C$$

Example (2): Evaluate the integral $\int \frac{(4x+5)dx}{(x^2-2x+2)^{\frac{3}{2}}}$

Solution: $I = \int \frac{(4x+5)dx}{(x^2-2x+2)^{\frac{3}{2}}} = \int \frac{(4x+5)dx}{(x^2-2x+1+1)^{\frac{3}{2}}} = \int \frac{(4x+5)dx}{((x-1)^2+1)^{\frac{3}{2}}}$

$$u = x - 1 \quad \to x = u + 1$$

$$du = dx$$

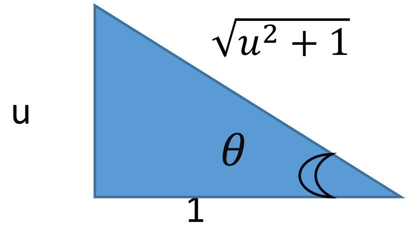
$$I = \int \frac{(4x+5)dx}{((x-1)^2+1)^{\frac{3}{2}}} = \int \frac{(4(u+1)+5)du}{((u)^2+1)^{\frac{3}{2}}} = \int \frac{(4u+9)du}{(u^2+1)^{\frac{3}{2}}}$$

$$= \int \frac{4udu}{(u^2+1)^{\frac{3}{2}}} + \int \frac{9\,du}{(u^2+1)^{\frac{3}{2}}} = 2\int 2u(u^2+1)^{\frac{-3}{2}}du + \int \frac{9\,du}{(u^2+1)^{\frac{3}{2}}}$$

$$= 2\frac{(u^2+1)^{\frac{-1}{2}}}{\frac{-1}{2}} + 9\int \frac{du}{(u^2+1)^{\frac{3}{2}}} = \frac{-4}{\sqrt{U^2+1}} + 9\int \frac{du}{(u^2+1)^{\frac{3}{2}}}$$

Consider $\int \frac{du}{(u^2+1)^{\frac{3}{2}}}$

Let $u = tan\theta \ \to du = sec^2\theta\, d\theta$



$$\int \frac{du}{(u^2+1)^{\frac{3}{2}}} = \int \frac{sec^2\theta d\theta}{(tan^2\theta+1)^{\frac{3}{2}}} = \int \frac{sec^2\theta d\theta}{(sec^2\theta)^{\frac{3}{2}}} = \int \frac{sec^2\theta d\theta}{(\mathbf{sec}^2\theta)(\mathbf{sec}^2\theta)^{\frac{1}{2}}}$$

$$= \int \frac{sec^2\theta d\theta}{(\mathbf{sec}^2\theta)(\sec\theta)} = \int \frac{1}{sec\theta}d\theta = \int cos\theta\, d\theta = sin\theta + C = \frac{u}{\sqrt{u^2+1}} + C$$

$$I = \frac{-4}{\sqrt{u^2+1}} + \frac{9u}{\sqrt{u^2+1}} + C = \frac{-4+9u}{\sqrt{u^2+1}} + C = \frac{-4+9(x-1)}{\sqrt{(x-1)^2+1}} + C = \frac{-4+9x-9}{\sqrt{x^2-2x+1+1}} + C = \frac{9x-13}{\sqrt{x^2-2x+2}} + C$$

Exercises :

(1) $\int_1^2 \frac{dx}{x^2+2x+5}$

(2) $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}$

(3) $\int \frac{\sqrt{x^2+2x}}{x+1}\,dx$

LECTURE (9)
CALCULUS 2

Integration of Rational Functions

المصادر: CALCULUS I
CALCULUS II

Integration of Rational Functions
تكاملات الدوال الكسرية النسبية(تجزئة كسور)

Definition: A rational function is a quotient of two polynomials, written as $R(X) = \frac{P_n(x)}{Q_m(x)}, Q_m(x) \neq 0$ where $P_n(x)$ and $Q_m(x)$ are polynomials Of degree *n* and *m* respectively.

(1) If $n \geq m$, we perform a long division until we obtain a rational function whose numerator degree less than to the denominator.

Example (1): Evaluate the integral $\int \frac{(x^2-3x+5)}{(x-2)}\, dx$

Solution: $\int \frac{(x^2-3x+5)}{(x-2)}\, dx = \int \left[(x-1) + \frac{3}{x-2}\right] dx$

$$= \frac{1}{2}x^2 - x + 3Ln|x-2| + C$$

$x - 1$

x-2 | $x^2 - 3x + 5$

$\underline{-x^2 + 2x}$

x+5

$\underline{\pm x \pm 2}$

3

Example (2): Evaluate the integral $\int \frac{x^2+2}{x^2+1}\, dx$

Solution: $\int \frac{x^2+2}{x^2+1}\, dx = \int \left(1 + \frac{1}{x^2+1}\right) dx$

$= x + tan^{-1}(x) + C$

$$\begin{array}{r|l} & 1 \\ x^2+1 & x^2+2 \\ & \underset{+}{-}x^2 \underset{+}{-}1 \\ & 1 \end{array}$$

(2) If $n < m$, we shall discuss the three cases of separating $\frac{P_n(x)}{Q_m(x)}$ into a sum of partial fractions.

Case (1): If the *m* factor of $Q_m(x)$ are all different and simple, that is , $Q_m(x) = (x-a_1)(x-a_2)\dots(x-a_m)$. Then we assign the sum of *m* partial fractions to these factors as follows $\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_m}{(x-a_m)}$ where $A_1, A_2, \dots, A_m$ are constants must be evaluated.

Example (3): Evaluate the integral $\int \frac{x^2+3x+3}{x^3-x}\,dx$

Solution: $\frac{x^2+3x+3}{x^3-x} = \frac{x^2+3x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$$= \frac{A(x-1)(x+1)+B(x)(x+1)+C(x)(x-1)}{x(x-1)(x+1)}$$

$x^2 + 3x + 3 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$

at $x = 0$ $\rightarrow 3 = A(0-1)(0+1) + 0 + 0 \rightarrow A = -3$

at $x = 1$ $\rightarrow 7 = 0 + B(1)(1+1) + 0 \rightarrow B = \frac{7}{2}$

at $x = -1$ $\rightarrow 1 = 0 + 0 + C(-1)(-1-1) \rightarrow C = \frac{1}{2}$

او طريقة اخرى

$x^2 + 3x + 3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$

$= (A + B + C)x^2 + (B - C)x - A$

$$\left.\begin{aligned} (A+B+C) &= 1 \\ B - C &= 3 \\ -A &= 3 \end{aligned}\right\} \quad A = -3\,,\ B = \frac{7}{2},\quad C = \frac{1}{2}$$

$$\therefore \int \frac{x^2+3x+3}{x^3-x}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right) dx = \int \left(\frac{-3}{x} + \frac{\frac{7}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\right) dx = -3\ln x + \frac{7}{2} ln(x-1) + \frac{1}{2} ln(x+1) + C$$

Case (2): Repeated factors of $Q_m(x)$ suppose $(x-a)^r$ is the highest power of $(x-a)$ which divides $Q_m(x)$.

Then we assign the sum of r partial fractions to these factors as follows:

$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_r}{(x-a)^r}$ where $A_1, A_2, \ldots, A_r$ are constants must be evaluated.

For example $\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+2)^3}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}+\frac{D}{(x+1)}+\frac{E}{(x+2)}+\frac{F}{(x+2)^2}+\frac{G}{(x+2)^3}$

Example (4): Evaluate the integral $\int\frac{(x+5)}{(x+2)(x-1)^2}\,dx$

Solution: $\frac{x+5}{(x+2)(x-1)^2}=\frac{A}{(x+2)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$

$$=\frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}$$

$x+5=A(x-1)^2+B(x+2)(x-1)+C(x+2)$

at $x=1 \rightarrow 6=0+0+3C \rightarrow C=2$

at $x=-2 \rightarrow 3=9A+0+0 \rightarrow A=\frac{1}{3}$

at $x=0 \rightarrow 5=\frac{1}{3}-2B+4 \rightarrow B=-\frac{1}{3}$

$$\int\frac{(x+5)}{(x+2)(x-1)^2}\,dx=\int\left[\frac{\frac{1}{3}}{x+2}-\frac{\frac{1}{3}}{x-1}+\frac{2}{(x-1)^2}\right]dx=\frac{1}{3}\ln(x+2)-\frac{1}{3}\ln(x-1)-\frac{2}{(x-1)}+C$$

Case (3): Quadratic factors of $Q_m(x)$ suppose $(x^2+ax+b)^r$ is the highest power of (x^2+ax+b) which divides $Q_m(x)$.

Then we assign the sum of r partial fractions to these factors as follows:

$$\frac{A_{1x+B_1}}{(x^2+ax+b)} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \cdots + \frac{A_rx+B_r}{(x^2+ax+b)^r}$$ where $A_1, A_2, \ldots, A_r$, $B_1, B_2, \ldots$, are constants must be evaluated.

For example $$\frac{x^2+2x-5}{x^2(x-1)(x^2+1)(x^2+2x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{Dx+E}{(x^2+1)} + \frac{Fx+G}{(x^2+2x+2)} + \frac{Hx+I}{(x^2+2x+2)^2}$$

Example (5): Evaluate the integral $\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx$

Solution:

$$\begin{array}{r|l} & x+1 \\ \hline x^4-2x^3+2x^2-2x+1 & x^5-x^4-3x+5 \\ & -\underline{x^5 \pm 2x^4 \mp 2x^3 \pm 2x^2 \mp x} \\ & x^4-2x^3+2x^2-4x+5 \\ & -\underline{x^4 \pm 2x^3 \mp 2x^2 \pm 2x \mp 1} \\ & -2x+4 \end{array}$$

$$\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \int \left(x+1+\frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\right)dx$$

$$= \frac{x^2}{2}+x+\int \frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\,dx$$

$$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

at $x = 1 \rightarrow 2 = 0 + 2B + 0 \rightarrow B = 1$

at $x = 0 \rightarrow 4 = A(-1)(1) + B(1) + (D)(-1)^2$

$4 = -A + B + D \quad \rightarrow \quad 3 = -A + D \quad \ldots(1)$

at $x = -1 \rightarrow 6 = A(-2)(-2) + 2B + (-C + D)(4)$

$\rightarrow 1 = -A - C + D \quad \ldots(2)$

at $x = 2 \rightarrow 0 = A(1)(5) + B(5) + (2C + D)(1)$

$\rightarrow -5 = 5A + 2C + D \quad \cdots(3)$

From (2) and(3) we have

$1 = -A - C + D \quad \ldots(2) \quad$ x 2

$-5 = 5A + 2C + D \quad \cdots(3)$

$2 = -2A - 2C + 2D \quad \ldots(2)$

$-5 = 5A + 2C + D \quad \cdots(3)$

$-3 = 3A + 3D \quad x \ \frac{1}{3} \quad \rightarrow -1 = A + D \quad \ldots(4)$

From (1) and(4) we have

$3 = -A + D \quad \ldots(1)$

$-1 = A + D \quad \ldots(4)$

$2D = 2 \quad \rightarrow D = 1$ and $A = -2$

From (2) we have $\quad C = 2$

$\therefore \quad A = -2, \quad B = 1, \quad C = 2, \quad D = 1.$

Let $\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\, dx = \frac{x^2}{2} + x + \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\right) dx$

$$= \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + \int \frac{2xdx}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + ln(x^2+1) + tan^{-1}x + C$$

$$= \frac{x^2}{2} + x + -2ln(x-1) - \frac{1}{x-1} + ln(x^2+1) + tan^{-1}x + C$$

Exercises :

(1) $\int \frac{x^4+1}{x^3-x} \, dx$.

(2) $\int \frac{x^2+2x+3}{(x+1)(x-1)(x-2)} \, dx$.

(3) $\int \frac{x^2+3x+3}{(x+1)(x^2-1)} \, dx$.

(4) $\int \frac{x^3-2x-5}{x^2(x-1)^2(x+1)^2} \, dx$.

(5) $\int \frac{x^3+3x^2-2x+1}{x^4+5x^2+4} \, dx$.

LECTURE (10)

CALCULUS 2

Integration of Irrational

and

Rational Trigonometric Functions

المصادر: CALCULUS I

CALCULUS II

الطريقة الثامنة في التكامل

<u>Integration of Irrational Functions:</u>

If the integral contain a single irrational expression of the form

$\sqrt[q]{(ax+b)} = (ax+b)^{\frac{1}{q}}$.

Let $z = (ax+b)^{\frac{1}{q}} \to \; z^q = ax+b \; \to qz^{q-1}dz = adx$

$\to dx = \frac{q}{a} z^{q-1} dz.$

Example (1): Evaluate the integral $\text{I} = \int \frac{2x+3}{\sqrt{x+2}}\, dx$

Solution: Let $z = (x+2)^{\frac{1}{2}} \to z^2 = x+2$

$\to x = z^2 - 2 \to 2zdz = dx$

$$\therefore \text{I} = \int \frac{2x+3}{\sqrt{x+2}}\, dx = 3\int \frac{2(z^2-2)+3}{z}\, 2zdz = 2\int (2z^2 - 1)\, dz$$

$$= 2\left(\frac{2z^3}{3} - z\right) + c = 2\left(\frac{2(x+2)^{\frac{3}{2}}}{3} - (x+2)^{\frac{1}{2}}\right) + c$$

Example (2): Evaluate the integral $I = \int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\, dx$

Solution: $I = \int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\, dx = \int \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{4}}} dx$

Let $z = x^{\frac{1}{4}} \rightarrow \quad z^4 = x \rightarrow \quad 4z^3 dz = dx$

$$I = \int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\, dx = \int \frac{z^2}{1+z} 4z^3 dz = 4\int \frac{z^5}{1+z} dz$$

$$= 4\int \left(z^4 - z^3 + z^2 - z + 1 - \frac{1}{z+1} \right) dz$$

$$= 4\left[\frac{1}{5}z^5 - \frac{1}{4}z^4 + \frac{1}{3}z^3 - \frac{1}{2}z^2 + z - ln(z+1) \right] + C$$

$$=4\left[\frac{1}{5}x^{\frac{5}{4}} - \frac{1}{4}x + \frac{1}{3}x^{\frac{3}{4}} - \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{4}} - ln\left(x^{\frac{1}{4}} + 1\right) \right] + C$$

Example (3): Evaluate the integral $I = \int \frac{\sqrt{4-x^2}}{x^3}\, dx$

Solution: $I = \int \frac{\sqrt{4-x^2}}{x^3}\, dx = \int \frac{(4-x^2)^{\frac{1}{2}}}{x^3}\, dx$

$z = (4-x^2)^{\frac{1}{2}} \quad \rightarrow z^2 = 4 - x^2 \quad \rightarrow x^2 = 4 - z^2$

$\rightarrow 2x\, dx = -2z dz \quad \rightarrow \quad x dx = -z\, dz$

$$I = \int \frac{\sqrt{4-x^2}}{x^3}\, dx = \int \frac{(4-x^2)^{\frac{1}{2}}}{x^3}\, dx = \int \frac{(4-x^2)^{\frac{1}{2}}}{x^4}\, x\, dx$$

$$= \int \frac{z}{(4-z^2)^2}(-z) dz = -\int \frac{z^2}{[(z+2)(z-2)]^2}\, dz = -\int \frac{z^2}{(z+2)^2(z-2)^2}\, dz$$

$$\frac{-z^2}{(z+2)^2(z-2)^2} = \frac{A}{(z+2)} + \frac{B}{(z+2)^2} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$A = \frac{1}{8}, \quad B = -\frac{1}{4}, \quad C = -\frac{1}{8}, \quad D = -\frac{1}{4}$

$$I = \int \left[\frac{\frac{1}{8}}{(z+2)} - \frac{\frac{1}{4}}{(z+2)^2} - \frac{\frac{1}{8}}{(z-2)} - \frac{\frac{1}{4}}{(z-2)^2} \right] dz = \frac{1}{8} ln(z+2) + \frac{1}{4}\frac{1}{(z+2)} - \frac{1}{8} ln(z-2) + \frac{1}{4}\frac{1}{(z-2)} + C$$

$$= \frac{1}{8} ln\left(\frac{z+2}{z-2}\right) + \frac{1}{4}\left[\frac{1}{z+2} + \frac{1}{z-2}\right] + C = \frac{1}{8} ln\left(\frac{z+2}{z-2}\right) + \frac{1}{4}\left[\frac{2z}{z^2-4}\right] + C = \frac{1}{8} ln\left(\frac{z+2}{z-2}\right) - \frac{1}{2}\left[\frac{z}{4-z^2}\right] + C$$

$$= \frac{1}{8} ln\left(\frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}-2}\right) - \frac{1}{2}\left(\frac{\sqrt{4-x^2}}{x^2}\right) + C$$

Exercises :

(1) $\int \frac{\sqrt{x}+2}{\sqrt{x}-1}\, dx$.

(2) $\int x\sqrt{x-1}\, dx$.

(3) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

(4) $\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}$.

(5) $\int \frac{x^5+2x^3}{\sqrt{x^2+4}}\, dx$.

الطريقة التاسعة في التكامل

<u>Integration of Rational Trigonometric Functions</u> :

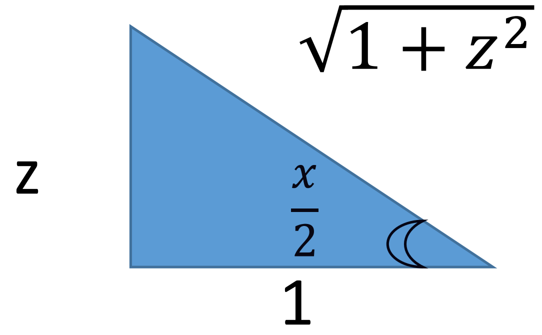
If the integral is a rational function of trigonometric, the substitution of $z = tan\frac{x}{2}$ will reduce the integral to rational function of which can be handle by method (7).

Let $z = \tan\frac{x}{2} \quad \rightarrow \frac{x}{2} = tan^{-1}z \quad \rightarrow \frac{dx}{2} = \frac{dz}{1+z^2} \quad \rightarrow dx = \frac{2dz}{1+z^2}$

$sin\frac{x}{2} = \frac{z}{\sqrt{1+z^2}}, \; cos\frac{x}{2} = \frac{1}{\sqrt{1+z^2}}$

$\sin x = 2\, sin\frac{x}{2} \; cos\frac{x}{2} = \frac{2z}{1+z^2}$

$\cos x = cos^2\frac{x}{2} - sin^2\frac{x}{2} = \frac{1-z^2}{1+z^2}$



Example (1): Evaluate the integral $I = \int \frac{dx}{5-4\cos x}$

Solution: Let $z = \tan\frac{x}{2}$

$$dx = \frac{2dz}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}, \quad \cos x = \frac{1-z^2}{1+z^2}$$

$$\therefore I = \int \frac{dx}{5-4\cos x} = \int \frac{\frac{2dz}{1+z^2}}{5-4\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{5(1+z^2)-4(1-z^2)}{\cancel{1+z^2}}}$$

$$= \int \frac{2dz}{5(1+z^2)-4(1-z^2)} = 2\int \frac{dz}{1+9z^2} = 2\int \frac{dz}{1+(3z)^2}$$

$$= \frac{2}{3}\tan^{-1} 3z + C = \frac{2}{3}\tan^{-1}\left[3\left(\tan\frac{x}{2}\right)\right] + C$$

Example (1): Evaluate the integral $I = \int \frac{dx}{3\cos x + 4\sin x}$

Solution: Let $z = \tan\frac{x}{2}$

$$dx = \frac{2dz}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}, \quad \cos x = \frac{1-z^2}{1+z^2}$$

$$\therefore\ I = \int \frac{dx}{3\cos x + 4\sin x} = \int \frac{\frac{2dz}{1+z^2}}{3\left(\frac{1-z^2}{1+z^2}\right) + 4\left(\frac{2z}{1+z^2}\right)} = \int \frac{2dz}{3(1-z^2)+8z}$$

$$= \int \frac{2dz}{3(1-z^2)+8z} = \int \frac{2dz}{3-3z^2+8z} = -2\int \frac{dz}{3z^2-8z-3} = -2\int \frac{dz}{(3z+1)(z-3)}$$

$$\frac{1}{(3z+1)(z-3)} = \frac{A}{(3z+1)} + \frac{B}{(z-3)} = \frac{A(z-3)+B(3z+1)}{(3z+1)(z-3)}$$

$$1 = A(z-3) + B(3z+1) \quad \rightarrow \quad A = -\frac{3}{10}, \quad B = \frac{1}{10}$$

$$I = -2\int \left[\frac{-\frac{3}{10}}{(3z+1)} + \frac{\frac{1}{10}}{(z-3)}\right] dz = \frac{-2}{-10}\int \frac{3dz}{(3z+1)} + \frac{-2}{10}\int \frac{dz}{(z-3)} = \frac{1}{5}\ln(3z+1) - \frac{1}{5}ln(z-3) + C$$

$$= \frac{1}{5}ln\left(\frac{3z+1}{z-3}\right) + C = \frac{1}{5}ln\left[\frac{3\left(\tan\frac{x}{2}\right)+1}{\left(\tan\frac{x}{2}\right)-3}\right] + C$$

Exercises :

(1) $\int \frac{dx}{2-\cos x}$.

(2) $\int \frac{dx}{3+2\cos x+2\sin x}$.

(3) $\int \frac{dx}{\tan x-\sin x}$.

(4) $\int_0^{\pi} \frac{dx}{1+\sin x}$.